Detailed derivation process of the FIK identity

Koji Fukagata
Department of Mechanical Engineering The University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113-8656, Japan
(Private note. Last update: December 3, 2004)

In the paper [Fukagata, Iwamoto, and Kasagi, *Phys. Fluids* **14**, L73-76 (2002)], we presented a relation of the contributions of the Reynolds stress distribution to the skin friction coefficient (FIK identity). The detailed derivation process from Eq. (8) to Eq. (10) in that paper, that was omitted for brevity, is presented here.

Equation (8) in Ref. 1 is

$$\frac{1}{8}C_f = \frac{\partial}{\partial y}\left[\overline{u'v'} - \frac{1}{Re_b}\frac{\partial \overline{u}}{\partial y}\right] + I''_x + \frac{\partial p''}{\partial x} + \frac{\partial \overline{u}}{\partial t} \, . \tag{1}$$

By integrating once from 0 to y, we obtain

$$\frac{1}{8}C_f y = \overline{u'v'} - \frac{1}{Re_b}\frac{\partial \overline{u}}{\partial y} + \frac{1}{Re_b}\frac{\partial \overline{u}}{\partial y}\bigg|_{y=0} + \int_0^y \left(I''_x + \frac{\partial p''}{\partial x} + \frac{\partial \overline{u}}{\partial t}\right) dy \, , \tag{2}$$

where $u'v'|_{y=0} = 0$ was used. By using Eq. (7) in Ref. 1, $(\partial u/\partial y)|_{y=0}/Re_b$ in the r.h.s. can be absorbed in the l.h.s. to read

$$\frac{1}{8}C_f (y-1) = \overline{u'v'} - \frac{1}{Re_b}\frac{\partial \overline{u}}{\partial y} + \int_0^y \left(I''_x + \frac{\partial p''}{\partial x} + \frac{\partial \overline{u}}{\partial t}\right) dy \, . \tag{3}$$

By integrating once more, we obtain

$$\frac{1}{8}C_f \left(\frac{y^2}{2} - y\right) = \int_0^y \overline{u'v'} dy - \frac{1}{Re_b}\overline{u} + \int_0^y \int_0^y \left(I''_x + \frac{\partial p''}{\partial x} + \frac{\partial \overline{u}}{\partial t}\right) dy dy \, , \tag{4}$$

where $\overline{u}_{y=0} = 0$ was used. Subsequently, an integration from 0 to 1 yields

$$-\frac{1}{24}C_f = \int_0^1 \int_0^y \overline{u'v'} dy dy - \frac{1}{2Re_b} + \int_0^1 \int_0^y \int_0^y \left(I''_x + \frac{\partial p''}{\partial x} + \frac{\partial \overline{u}}{\partial t}\right) dy dy dy \, , \tag{5}$$

where the definition of the bulk mean velocity (Ref. 1), i.e., $\int_0^1 \overline{u} dy = 1/2$, was used.

The double integration in Eq. (5) can be transformed into a single integration, i.e.,

$$\begin{aligned} \int_0^1 \int_0^y \overline{u'v'} dy dy &= \int_0^1 \left(1 \cdot \int_0^y \overline{u'v'} dy\right) dy \\ &= \left[y \int_0^y \overline{u'v'} dy\right]_0^1 - \int_0^1 y \, \overline{u'v'} dy \\ &= \int_0^1 (1-y)(\overline{u'v'}) dy \end{aligned} \tag{6}$$

Likewise, the triple integration can also be transformed into a single integration.

By substituting those into Eq. (5), we finally obtain Eq. (10) in Ref. 1, i.e.,

$$\frac{1}{2} = Re_b \left[\frac{C_f}{24} - \int_0^1 (1-y)(-\overline{u'v'}) \, dy + \frac{1}{2}\int_0^1 (1-y)^2 \left(I''_x + \frac{\partial p''}{\partial x} + \frac{\partial \overline{u}}{\partial t}\right) dy\right] \, . \tag{7}$$

References

[1] Fukagata, K., Iwamoto, K. and Kasagi, N., Contribution of Reynolds stress distribution to the skin friction inwall-bounded flows, *Phys. Fluids* **14**, L73-L76, (2002).